

Entanglement entropy in particle decay.

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Abstract

The decay of a parent particle into two or more daughter particles results in an entangled quantum state, as a consequence of conservation laws in the decay process. We use the Wigner-Weisskopf formalism to construct an approximation to this state that evolves in time in a *manifestly unitary* way. We then construct the entanglement entropy for one of the daughter particles by use of the reduced density matrix obtained by tracing out the unobserved states and follow its time evolution. We find that it grows over a time scale determined by the lifetime of the parent particle to a maximum, which when the width of the parent particle is narrow, describes the phase space distribution of maximally entangled Bell-like states.

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I. INTRODUCTION

Once described as the source of “spooky action at a distance” by Einstein, Podolsky and Rosen (EPR)[1], quantum entanglement has now come to be viewed as a resource to be exploited in a number of venues. It serves as the workhorse for quantum computations[2–8], as well as a way to measure a variety of quantities in particle physics, such as flavor entanglement in $\Upsilon(4S) \rightarrow B^0 \bar{B}^0$ via the analysis of the time dependence of the flavor asymmetry[9] and time reversal violation in the B^0 system[10, 11]. The possibility has also been advanced[12, 13] to use entanglement correlations to establish bounds on the $B_s - \bar{B}_s$ width difference and CP violating phases. Entanglement between the charged lepton and its associated neutrino in the decay of pseudoscalar mesons has been recently argued to play an important role in the coherence (and decoherence) aspects of neutrino oscillations[14–17] with potentially important corrections in short baseline oscillation experiments[18]. In condensed matter and quantum optics, the spontaneous decay of excited atomic states leads to quantum entangled states of photons and atoms or spin-qubits, which can then be exploited as platforms for quantum computing[3, 4].

Given a pure quantum system consisting of entangled subsystems, it may not be possible to measure the separate state of all of the subsystems (or while possible, we may opt *not* to measure them). We can then construct a *reduced* density matrix for the subsystem(s) we do measure by tracing over the allowed states of the unobserved subsystem(s). This then leads directly to the concept of entanglement entropy: this is the von Neumann entropy of the reduced density matrix. It reflects the loss of information that was originally present in the entangled state from the quantum correlations. This entanglement entropy has been the focus of several studies in statistical and quantum field theories where subsystems are spatially correlated across boundaries by tracing over the degrees of freedom of one part of the system[19–24] and have been extended to the case including black holes[25, 26], particle production in time dependent backgrounds[27] and cosmological space times[28]. Momentum space entanglement and renormalization has been recently studied in ref.[29].

While entanglement entropy has been mostly studied within the context of a quantum system subdivided by space-like regions (see references above), in this article we study the time evolution of the entanglement entropy in the ubiquitous case of particle decay. The construction of the relevant states in this case relies on the Wigner-Weisskopf theory of spontaneous emission[30, 31], which provides a *non-perturbative* method for obtaining the quantum state arising from spontaneous decay. The knowledge of the full quantum state can then be used to obtain the entanglement entropy contained in, for example, the photon-spin qubit correlations generated from the dynamics of spontaneous decay in solid state systems[32].

Recently this theory was generalized to relativistic quantum field theory to yield insight into the quantum states from particle decay in cosmology[33], and to describe potential decoherence effects in neutrino oscillations in short baseline experiments[17, 18].

Here we extend and generalize the Wigner-Weisskopf method discussed in[18, 32, 33] to describe particle decay in quantum field theory and apply it to the simple case of a bosonic parent particle decaying into two bosonic daughter particles, although we argue that the results are general. We address in detail the important aspect of unitarity and obtain the entanglement entropy by tracing over the degrees of freedom associated with an unobserved daughter particle. We show in detail how unitary time evolution yields an entanglement entropy that grows over the lifetime of the parent particle and saturates to the entropy of

maximally entangled states.

II. THE WIGNER-WEISSKOPF APPROXIMATION

Consider a system described by a total Hamiltonian that can be decomposed as $H = H_0 + H_i$, where H_0 is the free Hamiltonian and H_i is the interaction part. As usual, the time evolution of a state in the interaction picture is given by

$$i\frac{d}{dt}|\Psi(t)\rangle = H_I(t)|\Psi(t)\rangle \quad (\text{II.1})$$

where

$$H_I(t) = e^{iH_0t} H_i e^{-iH_0t} \quad (\text{II.2})$$

is the interaction Hamiltonian in the interaction picture. The formal solution of (II.1) is given by

$$|\Psi(t)\rangle = U(t, t_0)|\Psi(t_0)\rangle \quad ; \quad U(t, t_0) = T(e^{-i\int_{t_0}^t H_I(t')dt'}) \quad (\text{II.3})$$

Expanding the state $|\Psi(t)\rangle$ in the basis of eigenstates of H_0 we have

$$|\Psi(t)\rangle = \sum_n C_n(t)|n\rangle \quad (\text{II.4})$$

where $H_0|n\rangle = E_n|n\rangle$. From eq.(II.1), upon expanding in basis states $|n\rangle$, it follows that

$$\frac{dC_n(t)}{dt} = -i \sum_m \langle n|H_I(t)|m\rangle C_m(t). \quad (\text{II.5})$$

This is an infinite set of differential equations that can be solved hierarchically leading in general to integro-differential equations at any given order in the perturbative expansion. Progress can be made by considering the evolution of the states that are coupled at a given order in perturbation theory and solving the coupled equations for these, thereby truncating the hierarchy at a given order in the perturbative expansion.

For the case of interest here, consider an initial value problem in which the system is prepared at an initial time $t = 0$ in a state $|A\rangle$, so that $C_A(0) = 1$, $C_{n \neq A} = 0$ and that the Hamiltonian H_I couples these states to a set of states $|\kappa\rangle$. Then, we can close the hierarchy at second order in the interaction by keeping only the coupling of the states $|A\rangle \leftrightarrow |\kappa\rangle$, i.e.

$$\frac{d}{dt} C_A(t) = -i \sum_{\kappa} \langle A|H_I(t)|\kappa\rangle C_{\kappa}(t) \quad (\text{II.6})$$

$$\frac{d}{dt} C_{\kappa}(t) = -i \langle \kappa|H_I(t)|A\rangle C_A(t). \quad (\text{II.7})$$

Using the initial conditions we obtain

$$C_{\kappa}(t) = -i \int_0^t dt' \langle \kappa|H_I(t')|A\rangle C_A(t'), \quad (\text{II.8})$$

which when inserted into (II.6) leads to

$$\frac{d}{dt}C_A(t) = - \int_0^t dt' \sum_{\kappa} \langle A|H_I(t)|\kappa\rangle \langle \kappa|H_I(t')|A\rangle C_A(t') = - \int_0^t dt' \Sigma_A(t-t')C_A(t'), \quad (\text{II.9})$$

where the second order self energy has been introduced

$$\Sigma_A(t-t') \equiv \sum_{\kappa} \langle A|H_I(t)|\kappa\rangle \langle \kappa|H_I(t')|A\rangle = \sum_{\kappa} \left| \langle A|H_I(0)|\kappa\rangle \right|^2 e^{i(E_A-E_{\kappa})(t-t')}. \quad (\text{II.10})$$

Higher order corrections can be included by enlarging the hierarchy, i.e. by considering the equations that couple the states κ to other states κ' via the Hamiltonian. The coefficients for the states κ' can be obtained by integration and can be inserted back in the equations for the coefficients C_{κ} (which already include their coupling to $|A\rangle$). Then formally integrating the equation and inserting the results back into the equation for $|A\rangle$ generates higher order corrections to the self-energy Σ_A . Finally, solving for the time evolution of C_A allows us to obtain the time evolution of the other coefficients.

As seen by the procedure described above, the Wigner-Weisskopf approach can be used to construct an approximate version of the quantum state in the presence of interactions. However, what is not altogether obvious is that the truncation of states used to construct the state gives rise to a state whose time evolution is *unitary*. This will be extremely important in the sequel since we will want to follow that time evolution of the entanglement entropy this state would provide after tracing out an unobserved subsystem as discussed in the introduction. We will need to be sure that there are no spurious effects in this evolution due to an approximation to the state.

The statement of unitarity is one of conservation of probability. From the evolution equation (II.5) and its complex conjugate it follows that

$$\frac{d}{dt} \sum_n |C_n(t)|^2 = -i \sum_{m,n} \left[C_m(t) C_n^*(t) \langle n|H_I(t)|m\rangle - C_n(t) C_m^*(t) \langle m|H_I(t)|n\rangle \right] = 0, \quad (\text{II.11})$$

as can be seen by relabeling $m \leftrightarrow n$ in the second term. Therefore $\sum_n |C_n(t)|^2 = \text{constant}$. Now this is an *exact* result; the question is whether and how is it fulfilled in the Wigner-Weisskopf approximation obtained by truncating the hierarchy to the set of equations (II.6,II.7).

Using (II.8, II.10)) consider

$$\sum_{\kappa} |C_{\kappa}(t)|^2 = \int_0^t dt_1 C_A^*(t_1) \int_0^t dt_2 \Sigma_A(t_1, t_2) C_A(t_2) \quad (\text{II.12})$$

inserting $1 = \Theta(t_1 - t_2) + \Theta(t_2 - t_1)$ in the time integrals it follows that

$$\begin{aligned} \sum_{\kappa} |C_{\kappa}(t)|^2 &= \int_0^t dt_1 C_A^*(t_1) \int_0^{t_1} dt_2 \Sigma_A(t_1, t_2) C_A(t_2) \\ &\quad + \int_0^t dt_2 C_A(t_2) \int_0^{t_2} dt_1 \Sigma_A(t_1, t_2) C_A^*(t_1) \end{aligned} \quad (\text{II.13})$$

using $\Sigma_A(t_1, t_2) = \Sigma_A^*(t_2, t_1)$ relabelling $t_1 \leftrightarrow t_2$ in the second line of (II.13) and using (II.9) we find

$$\sum_{\kappa} |C_{\kappa}(t)|^2 = - \int_0^t dt_1 \left[C_A^*(t_1) \frac{d}{dt_1} C_A(t_1) + C_A(t_1) \frac{d}{dt_1} C_A^*(t_1) \right] = - \int_0^t dt_1 \frac{d}{dt_1} |C_A(t)|^2 = 1 - |C_A(t)|^2 \quad (\text{II.14})$$

where we have used ($C_A(0) = 1$). This is the statement of unitary time evolution, namely

$$|C_A(t)|^2 + \sum_{\kappa} |C_{\kappa}(t)|^2 = |C_A(0)|^2 \quad (\text{II.15})$$

This is an important result. As we will see below, standard perturbative calculations of this state would *not* yield a state that evolved unitarily. The Wigner-Weisskopf state involves a non-perturbative dressing up of the state and despite its approximate nature, this dressing up captures the physics to a sufficient extent to guarantee unitary time evolution.

III. REDUCED DENSITY MATRIX AND ENTANGLEMENT ENTROPY.

Now we are ready to turn to the problem we really want to consider: the state that appears after the decay of a parent particle Φ into two daughters χ, ψ . For simplicity and to highlight the main concepts we treat all fields as bosonic massive fields with masses m_{Φ} and m_{χ}, m_{ψ} respectively. We consider a typical interaction vertex described by the interaction Hamiltonian

$$H_i = g \int d^3x \Phi(\vec{x}) \chi(\vec{x}) \psi(\vec{x}). \quad (\text{III.1})$$

We quantize the fields in a volume V , so that in the interaction picture they can be written as:

$$\Psi(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{2E_k V}} \left[a_{\vec{k}} e^{-iE_k t} e^{i\vec{k} \cdot \vec{x}} + a_{\vec{k}}^{\dagger} e^{iE_k t} e^{-i\vec{k} \cdot \vec{x}} \right] ; \quad \Psi = \Phi, \chi, \psi. \quad (\text{III.2})$$

For the case of interest here, namely the decay process $\Phi \rightarrow \chi \psi$ we consider that the initial state is a single particle Φ at rest, therefore the initial condition is $C_{\Phi}(\vec{k} = 0; t = 0) = 1$, $C_{\kappa} = 0$ for $|\kappa\rangle \neq |1_0^{\Phi}\rangle$. The interaction Hamiltonian (III.1) connects the initial state, $|1_0^{\Phi}\rangle$ to the states, $|\kappa\rangle = |\chi_{-\vec{p}}\rangle \otimes |\psi_{\vec{p}}\rangle$. These states in turn are coupled back to $|1_0^{\Phi}\rangle$ via H_I , these processes are depicted in fig.(1).

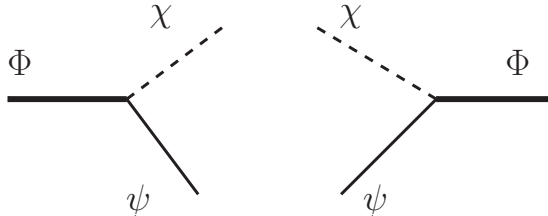


FIG. 1: Transitions $|\Phi\rangle \leftrightarrow |\chi\rangle|\psi\rangle$ up to order g^2 that determine Σ_{Φ} .

Thus to leading order in g we find from the intermediate states shown in fig.(1)

$$\Sigma_{\Phi}(t-t') = \sum_{\vec{p}} |\langle 1_{\vec{0}}^{\Phi} | \hat{H}_I(0) | \chi_{-\vec{p}}, \psi_{\vec{p}} \rangle|^2 e^{i(m_{\Phi}-E_{\chi}(p)-E_{\psi}(p))(t-t')} . \quad (\text{III.3})$$

The interaction Hamiltonian also connects a single Φ -particle state to an intermediate state with three other particles and this state back to the single Φ particle state yielding a disconnected contribution to the self energy depicted in fig.(2). This contribution is just a renormalization of the vacuum energy and only contributes to an overall phase that multiplies the single particle Φ state and will be neglected in the following analysis. For a more detailed discussion of this contribution see ref.[33].

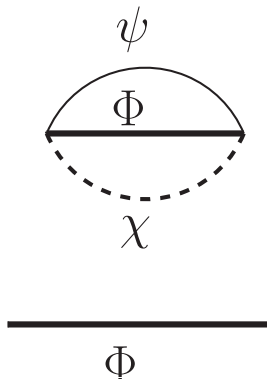


FIG. 2: Order g^2 correction to the vacuum energy. Yields an overall phase for the quantum state $|1^{\Phi}\rangle$ [33].

At higher order in g there are higher order contributions to the Φ self energy from other multiparticle states. However, we will only be considering states connected to $|1^{\Phi}\rangle$ to first order in perturbation theory. As discussed in detail in refs.[18, 33] the Wigner-Weisskopf method provides a *non-perturbative resummation* of self-energies in *real time* akin to the dynamical renormalization group[34]. The self-energy eq.(III.3) is recognized as the one-loop retarded self energy of the field Φ with the $|\chi\rangle|\psi\rangle$ intermediate state[18, 33].

Solving eq.(II.9) produces a solution for the time evolution of the Φ amplitude. We can use the solution for $C_{\Phi}(t)$ to obtain an expression for the amplitudes $C_{\kappa}(t)$ which allows for computation of the probability of occupying a particular state at any given time. We may solve eq.(II.9) either via Laplace transform, or in the case of weak coupling, a derivative expansion which yields the same result at long times ($t \gg 1/m_{\Phi}$). In ref.[33] the equivalence between the two approaches is discussed in detail. Here, we follow the latter method which is the original Wigner-Weisskopf approximation[30, 31].

We begin by defining the quantity

$$W_0(t, t') = \int_0^{t'} dt'' \Sigma_{\Phi}(t - t'') \quad (\text{III.4})$$

so that

$$\frac{d}{dt'} W_0(t, t') = \Sigma_{\Phi}(t - t') \quad , \quad W_0(t, 0) = 0 \quad (\text{III.5})$$

Integrating eq.(II.9) by parts yields

$$\frac{d}{dt}C_\Phi(t) = - \int_0^t dt' \Sigma_\Phi(t-t') C_\Phi(t') = -W_0(t,t) C_\Phi(t) + \int_0^t dt' W_0(t,t') \frac{d}{dt'} C_\Phi(t'). \quad (\text{III.6})$$

The first is term second order in H_I whereas the second term is of fourth order in H_I and will be neglected. This approximation is equivalent to the Dyson resummation of the one-loop self energy diagrams. Thus to leading order, eq.(II.9) becomes

$$\frac{d}{dt}C_\Phi(t) + W_0(t,t)C_\Phi(t) = 0, \quad (\text{III.7})$$

where

$$W_0(t,t) = \int_0^t dt' \Sigma_\Phi(t-t') = \int_0^t dt' \sum_{\vec{p}} |\langle 1_0^\Phi | \hat{H}_I(0) | \chi_{-\vec{p}}, \psi_{\vec{p}} \rangle|^2 e^{i(m_\Phi - E_\chi(p) - E_\psi(p))(t-t')} \quad (\text{III.8})$$

Inserting a convergence factor and taking the limit $t \rightarrow \infty$ consistently with the Wigner-Weisskopf approximation, we find¹

$$W_0(t,t) = \lim_{\epsilon \rightarrow 0^+} i \sum_{\kappa} \frac{|\langle 1_0^\Phi | \hat{H}_I(0) | \chi_{-\vec{p}}, \psi_{\vec{p}} \rangle|^2}{m_\Phi - E_\chi(p) - E_\psi(p) + i\epsilon} = i\Delta E_\Phi + \frac{\Gamma}{2} \quad (\text{III.9})$$

where

$$\Delta E_\Phi \equiv \mathcal{P} \sum_{\vec{p}} \frac{|\langle 1_0^\Phi | \hat{H}_I(0) | \chi_{-\vec{p}}, \psi_{\vec{p}} \rangle|^2}{m_\Phi - E_\chi(p) - E_\psi(p)}, \quad (\text{III.10})$$

is the second order shift in the energy which will be absorbed into a renormalization of the Φ mass and

$$\Gamma \equiv 2\pi \sum_{\vec{p}} |\langle 1_0^\Phi | \hat{H}_I(0) | \chi_{-\vec{p}}, \psi_{\vec{p}} \rangle|^2 \delta(m_\Phi - E_\chi(p) - E_\psi(p)) \quad (\text{III.11})$$

is the decay width as per Fermi's Golden rule. Therefore in this approximation, we arrive at

$$C_\Phi(t) = e^{-i\Delta E_\Phi t} e^{-\frac{\Gamma}{2}t}, \quad (\text{III.12})$$

where we now consider a Φ with $\vec{k} = 0$ (decay at rest) and find

$$\mathcal{M}_\Phi(p) = \langle 1_0^\Phi | \hat{H}_I(0) | \chi_{-\vec{p}}, \psi_{\vec{p}} \rangle = \frac{g}{\sqrt{8Vm_\Phi E_\chi(p)E_\psi(p)}} \quad (\text{III.13})$$

leading to

$$\Gamma = 2\pi \sum_p |\mathcal{M}_\Phi(p)|^2 \delta(m_\Phi - E_\chi(p) - E_\psi(p)) = \frac{g^2 p^*}{8\pi m_\Phi^2} \quad (\text{III.14})$$

¹ The long time limit in the Wigner-Weisskopf approximation is equivalent to the Breit-Wigner approximation of a resonant propagator[33] and holds for $t \gg 1/m_\Phi$.

where

$$p^* = \frac{1}{2m_\Phi} \left[m_\Phi^4 + m_\chi^4 + m_\psi^4 - 2m_\Phi^2 m_\chi^2 - 2m_\Phi^2 m_\psi^2 - 2m_\chi^2 m_\psi^2 \right]^{1/2}. \quad (\text{III.15})$$

Inserting (III.12) into (II.8) we find the quantum state

$$|\Psi(t)\rangle = e^{-i\Delta E_\Phi t} e^{-\frac{\Gamma}{2}t} |1_0^\Phi; 0_\chi; 0_\psi\rangle + \sum_{\vec{p}} \mathcal{C}(p; t) |\chi_{-\vec{p}}\rangle |\psi_{\vec{p}}\rangle |0_\Phi\rangle \quad (\text{III.16})$$

where

$$\mathcal{C}(p; t) = \mathcal{M}_\Phi(p) \frac{\left[1 - e^{-i(m_{\Phi,R} - E_\chi(p) - E_\psi(p) - i\Gamma/2)t} \right]}{\left(E_\chi(p) + E_\psi(p) - m_{\Phi,R} + i\Gamma/2 \right)} \quad (\text{III.17})$$

where $m_{\Phi,R} = m_\Phi + \Delta E_\Phi$ is the renormalized mass of the Φ particle. In what follows we drop the subscript R and always refer to the renormalized mass.

At this stage we can make contact with the momentum entanglement discussion in ref.[29] and at the same time exhibit the true non-perturbative nature of the results above by considering the state obtained using naive perturbation theory.

Taking the initial state at $t = 0$ to be $|1_0^\Phi\rangle$, then to leading order in g , the time evolved state is given by

$$|\Psi(t)\rangle = \left[1 - i \int_0^t e^{iH_0 t'} H_i e^{-iH_0 t'} dt' + \dots \right] |1_0^\Phi\rangle. \quad (\text{III.18})$$

Introducing a resolution of the identity $1 = \sum_\kappa |\kappa\rangle \langle \kappa|$ we find

$$|\Psi(t)\rangle = |1_0^\Phi\rangle + \sum_{\vec{p}} \left[\mathcal{M}_\Phi(p) \left(\frac{1 - e^{-i(m_\Phi - E_\chi(p) - E_\psi(p))t}}{E_\chi(p) + E_\psi(p) - m_\Phi} \right) |\chi_{-\vec{p}}\rangle |\psi_{\vec{p}}\rangle |0_\Phi\rangle + \dots \right] \quad (\text{III.19})$$

In this expression we have neglected a disconnected three particle intermediate state in which the initial $|1_0^\Phi\rangle$ remains and the interaction creates an intermediate state with three particles. This contribution is truly perturbative, does not contain resonant denominators as (III.19) and corresponds to the disconnected diagram in fig.(2).

The probability of finding the daughter states is given by the familiar result (Fermi's Golden rule)

$$\mathcal{P}(t) = \sum_{\vec{p}} |\mathcal{M}_\Phi(p)|^2 \left[\frac{\sin((m_\Phi - E_\chi(p) - E_\psi(p))t/2)}{(m_\Phi - E_\chi(p) - E_\psi(p))} \right]^2 = \Gamma t \quad (\text{III.20})$$

where Γ is given by (III.14). This result is obviously only valid for $t \ll 1/\Gamma$. It is now clear that the generalized Wigner-Weisskopf method that yields the state (III.16) with the coefficients given by (III.17) is truly non-perturbative.

The momentum entanglement between the daughter particles is akin to that discussed perturbatively in ref.[29] with some important differences. In ref.[29] momentum entanglement was studied for the vacuum wave function. The corresponding contributions are truly perturbative and do not feature the resonant denominators that lead to secular growth in time and are similar to the contributions that we neglect and are described by fig.(2).

The quantum state (III.16) describes an *entangled* state of the parent and daughter particles. The full (pure state) density matrix is given by

$$\rho(t) = |\Psi(t)\rangle\langle\Psi(t)| \quad (\text{III.21})$$

and its trace is given by

$$\text{Tr}\rho(t) = e^{-\Gamma t} + V \int \frac{d^3p}{(2\pi)^3} |\mathcal{C}(p; t)|^2 \quad (\text{III.22})$$

The momentum integral in (III.22) can be computed by changing variables to $\mathcal{E} = E_\chi(p) + E_\psi(p)$. In the narrow width limit $\Gamma \ll m_\Phi, m_\chi + m_\psi$, the integrand is sharply peaked at $\mathcal{E} = m_\Phi$ so that the lower limit can be consistently taken to $-\infty$ thus allowing the integral can be computed by contour integration. We find

$$V \int \frac{d^3p}{(2\pi)^3} |\mathcal{C}(p; t)|^2 = 1 - e^{-\Gamma t}, \quad (\text{III.23})$$

confirming that

$$\text{Tr}\rho(t) = 1, \quad (\text{III.24})$$

consistent with unitary time evolution and the unitarity relation (II.15). Furthermore the average number of ψ (or χ) particles is given by

$$n^\psi(p; t) = \langle\Psi(t)|a_\psi^\dagger(p)a_\psi(p)|\Psi(t)\rangle \equiv (2\pi)^3 \frac{dN^\psi(t)}{d^3x d^3p} = |\mathcal{C}(p; t)|^2, \quad (\text{III.25})$$

thus the *total* number of ψ (or χ) particles is

$$N^\psi(t) = V \int \frac{d^3p}{(2\pi)^3} n^\psi(p; t) = 1 - e^{-\Gamma t}. \quad (\text{III.26})$$

Tracing out one of the daughter particles, for example χ if it is unobservable, leads to a *reduced density matrix*

$$\rho_\psi(t) = \text{Tr}_\chi \rho(t) = e^{-\Gamma t} |1_0^\Phi\rangle\langle 1_0^\Phi| + \sum_{\vec{p}} |\mathcal{C}(p; t)|^2 |\psi_{\vec{p}}\rangle\langle\psi_{\vec{p}}|. \quad (\text{III.27})$$

This reduced density matrix describes a statistical mixture of states, the Von-Neumann entanglement entropy is therefore given by

$$S(t) = -n^\Phi(0, t) \ln[n^\Phi(0, t)] - \sum_{\vec{p}} n^\psi(p; t) \ln[n^\psi(p; t)]. \quad (\text{III.28})$$

where $n^\psi(p)$ is given by (III.25) and $n^\Phi(0, t) = e^{-\Gamma t}$. Because in the narrow width limit $|\mathcal{C}(p; t)|^2$ is a sharply peaked distribution, under integration with functions that vary smoothly near $p \simeq p^*$ it can be replaced by

$$n^\psi(p; t) = |\mathcal{C}(p; t)|^2 \simeq \frac{2\pi^2 m_\Phi [1 - e^{-\Gamma t}]}{V p^* E_\chi(p) E_\psi(p)} \frac{1}{2\pi} \frac{\Gamma}{(E_\chi(p) + E_\psi(p) - m_\Phi)^2 + (\Gamma/2)^2}. \quad (\text{III.29})$$

Using this approximation we find

$$S(t) = \Gamma t e^{-\Gamma t} - [1 - e^{-\Gamma t}] \ln [1 - e^{-\Gamma t}] - [1 - e^{-\Gamma t}] \ln [n^\psi(p^*; \infty)] \quad (\text{III.30})$$

where

$$n^\psi(p^*; \infty) = \frac{4\pi m_\Phi}{V p^* E_\chi(p^*) E_\psi(p^*) \Gamma}. \quad (\text{III.31})$$

This result has the following interpretation. In the asymptotic limit $t \gg 1/\Gamma$, the entanglement (von Neumann) entropy approaches (minus) the logarithm of the available states. The decay of the parent particle produces entangled pairs in which each member features a very narrow distribution centered at p^* of width $\sim \Gamma$ and height $\sim 1/\Gamma$. The total area in momentum space yields $1/V$ since there is only one particle (of either type) produced in the volume V . Within the range of momenta centered at p^* and of width Γ all of the available single particle states have equal probability $\propto 1/V$, therefore these states are *maximally entangled* as Bell states. This observation becomes clearer recognizing that a typical quantum state that contributes to the sum in (III.16) is of the form

$$\mathcal{C}(p^*, t) \left[|\chi(\vec{p}^*)\rangle |\psi(-\vec{p}^*)\rangle + |\chi(-\vec{p}^*)\rangle |\psi(\vec{p}^*)\rangle \right]; \quad \vec{p}^* = p^* \vec{n}, \quad (\text{III.32})$$

where \vec{n} is the direction of emission of either member of the pair. The quantum states with momenta $p^* - \Gamma/2 \leq p \leq p^* + \Gamma/2$ are represented with nearly the same probability $|\mathcal{C}(p^*, t)|^2 \propto 1/V\Gamma$ in the sum. These states are Bell-type states and are maximally entangled, in fact these are similar, up to an overall normalization, to the entangled $B^0 \bar{B}^0$ states resulting from the decay of the $\Upsilon(4S)$ resonance[9–11], but with the opposite relative sign because of charge conjugation. If the decaying particle has a short lifetime corresponding to a broad resonance, the emitted pairs will feature a distribution of momenta with probabilities $|\mathcal{C}(p, t)|$ determined by the Lorentzian profile of the resonance.

As discussed above $n^\psi(p^*; \infty)$ (see eqn. (III.25)) is the asymptotic phase space density of the produced particle (either χ or ψ). The entanglement entropy vanishes at the initial time since the density matrix at $t = 0$ is a pure state and grows to its asymptotic value on a time scale $1/\Gamma$.

The volume factor requires further discussion. It appears as a result of treating the initial (and final) states as plane waves. A more physical description of the decay process would treat the initial state as a wave packet localized in space. A natural localization length scale for the decaying state would be the decay length $\tau \simeq 1/\Gamma$; for example, for the experimentally relevant case of a pion decaying at rest, $\tau \simeq 7.8$ mts is also approximately the typical experimental scale of the production region. In particular, for a pion decaying into a muon and a neutrino with $p^* = 30$ MeV and $m_\nu \ll m_\mu$ and $V \sim 1/\Gamma^3$, we find $n^\mu(p^*; \infty) \simeq 10^{-30}$.

IV. CONCLUSIONS AND FURTHER QUESTIONS

The decay of a parent particle leads to a quantum entangled state of the daughter particles as a consequence of conservation laws in the decay process. We generalized and extended a method used in the study of spontaneous decay of atomic systems to the realm of quantum field theory to obtain in a consistent approximation, the full quantum state that describes the time evolution of the decaying particle and the production of the daughter particles. This

method is non-perturbative and is manifestly unitary. We have implemented the method to study the simpler case of bosonic parent and daughter particles to highlight the main concepts and consequences, however, the results are quite general.

The full quantum state resulting from the time evolution of the decaying particle yields a pure state density matrix. However, if one (or more) daughter particles is unobserved, tracing over their degrees of freedom leads to a mixed state density matrix whose time evolution is completely determined by the *unitary* time evolution of the decay process. This mixed state density matrix features an *entanglement entropy* which is a manifestation of the quantum correlations of the entangled product state. We obtained the time evolution of the entanglement entropy and show that it grows on a time scale determined by the lifetime of the decaying particle and reaches a maximum that corresponds to the logarithm of the available phase space states of the decay particles. For a parent particle described by a narrow resonance the distribution of produced (entangled) daughter particles is nearly a constant in a narrow energy-momentum region, the emitted particles are nearly maximally entangled Bell-states. These concepts *may* prove relevant in the statistical analysis of the time evolution of entangled B-meson pairs in studies of CP violation.

This work can be viewed as prelude to the calculation of the entanglement entropy due to particle decays in de Sitter space[33]. There, due to the fact that particle can decay into itself with momenta that are much less than the Hubble constant H_{deSitter} [33], we expect there to be an interplay between the horizon size and the decay rate that will feed in to the behavior of the entanglement entropy. This may then mix the ideas of entanglement entropy developed here with those coming from studies of spatially separated portions of de Sitter space such as in ref.[28]. Work on these aspects will be reported elsewhere[35].

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